Name: RED ID:

Spring 2024 Math 245 Final Exam

Please read the following directions:

Please write legibly, with plenty of white space. Please **print** your name and REDID in the designated spaces above. Please fit your answers into the designated areas; material outside the designated areas (such as on this cover page) will not be graded. To get credit, you must also show adequate work to justify your answers. If unsure, show the work. The first four questions are worth 7-14 points, and the remaining sixteen questions are worth 9-18 points. The maximum possible score is $4 \times 14 + 16 \times 18$, for a total of 344 points. You will also get one extra point for free, just because I love you! The use of notes, books, calculators, or other materials on this exam is strictly prohibited, except you may bring one 8.5"x11" page (both sides) with your handwritten notes. If you need scratch paper, you may use any blank space on your note sheet and on this front page. This exam will begin at 10:30 and will end at 12:30; pace yourself accordingly. Good luck!

Special exam instructions for SSW-1500:
1. Please stow all bags/backpacks/boards at the front of the room. All contraband, except phones, must be stowed in your bag. All smartwatches and phones must be silent, non-vibrating, and either in your pocket or stowed in your bag.
2. Please remain quiet to ensure a good test environment for others.
3. Please keep your exam on your desk; do not lift it up for a better look.
4. If you have a question or need to use the restroom, please come to the front. Bring your exam. I cannot come to you unless you are sitting by an aisle.
5. If you are done and want to submit your exam and leave, please wait until one of the designated exit times, listed below. Please do **NOT** leave at any other time. If you are sure you are done, just sit and wait until the next exit time, with this cover sheet visible.

Designated exam exit times:
- 10:50 "See you next semester (or this summer: class runs May 20 - August 8)"
- 11:10 "I wish I had studied more"
- 11:30 "One extra hour of drinking – worth it"
- 11:50 "Maybe this will be good enough"
- 12:10 "There is nothing more in my brain, let me out of here"
- 12:30 "I need every second I can get"

Problem 1. Carefully state the following definitions:

a. contrapositive

b. set difference

Problem 2. Carefully state the following definitions:

a. reflexive closure

b. chain

Problem 3. Carefully state the following definitions:

a. image

b. injection

Problem 4. Carefully state the following theorems:

a. Cantor's Theorem

b. Zorn's Lemma

Problem 5. Let $a, b \in \mathbb{N}_0$ with $a > b$. Prove that $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$.

Problem 6. Carefully state and prove the Disjunctive Syllogism Theorem.

Problem 7. Let p, q, r be arbitrary propositions. Without using truth tables, prove that $p \to q, r \to p \vdash r \to q$.

Problem 8. Prove or disprove: $\forall n \in \mathbb{N},\ !m \in \mathbb{N},\ n = m^2$.

Problem 9. Let $a_n = 3 + \sin n + \cos n$. Prove or disprove that $a_n = O(1)$.

Problem 10. Prove that, for every $n \in \mathbb{N}$, we have $\sum_{i=n}^{2n} 2^i = 2^n(2^{n+1} - 1)$.

Problem 11. Let S, T be sets. Suppose that $S \Delta T = S$. Prove that $T = \emptyset$.

Problem 12. Prove or disprove: For all finite sets A, B, C, if $|A| < |B|$ then $|A \times C| < |B \times C|$.

Problem 13. Let U be a set (the universe) and consider $f : 2^U \to 2^U$ given by $f : S \mapsto S^c$. You may assume that f is a function. Prove or disprove that f is a bijection.

Problem 14. Find all integers $x \in [0, 36)$ satisfying $15x \equiv 10 \pmod{36}$.

Problem 15. Find a partial order on $\mathbb{N}$ in which $2 < 1$, $3 < 1$, and $2 \| 3$.

Problem 16. Consider the function $f : \mathbb{N} \to \mathbb{Z}$ given by $f(n) = \begin{cases} n/2 & n \text{ even} \\ -(n-1)/2 & n \text{ odd} \end{cases}$. Prove or disprove that f is injective.

For problems 17-20, we let $S = \mathbb{Z}/\equiv_5$, a set of equivalence classes. Except where otherwise indicated, $[a]$ denotes an equivalence class of the $\equiv_5$ modular equivalence relation, i.e. $[a] \in S$.

Problem 17. Consider a relation on S, given by $R_1 = \{([x],[y]) : 0 \in [x+y]\}$. Draw the digraph representing R_1, and prove or disprove that R_1 is reflexive.

Problem 18. Consider again $R_1 = \{([x],[y]) : 0 \in [x+y]\}$, a relation on S. Prove or disprove that R_1 is a function.

Problem 19. Consider a relation on S, given by $R_2 = \{([x],[y]) : [x^2] = [y^2]\}$. Draw the digraph representing R_2, and prove or disprove that R_2 is right total.

Problem 20. Consider again $R_2 = \{([x],[y]) : [x^2] = [y^2]\}$, a relation on S. Prove that R_2 is an equivalence relation, and calculate $[\ [1]\ ]_{R_2}$ explicitly.